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But the first three terms of the series,  $u_0, u_1, u_2$ , &c., are  $u_0 = 1, u_1 = 1$ , and  $u_2 = 2$ . Hence, putting  $x = 0$ , we have  $A u_1 = 1$ , and  $P u_0 = 1$ ,  $\therefore A = -1$  and  $P = -1$ .

$$\therefore z^2 - z = 1; \therefore r_1 = \frac{1 + \sqrt{5}}{2} \text{ and } r_2 = \frac{1 - \sqrt{5}}{2}.$$

$$\therefore u_x = C_1 \left( \frac{1 + \sqrt{5}}{2} \right)^x + C_2 \left( \frac{1 - \sqrt{5}}{2} \right)^x.$$

When  $x = 0$   $u_0 = C_1 + C_2 = 1$ . And when

$$x = 1 \quad u_1 = C_1 \left( \frac{1 + \sqrt{5}}{2} \right) + C_2 \left( \frac{1 - \sqrt{5}}{2} \right) = 1.$$

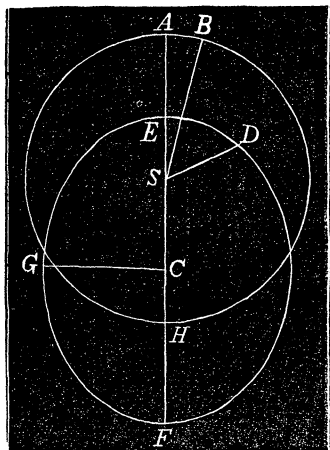
$$\therefore C_1 = \frac{5 + \sqrt{5}}{10}, \text{ and } C_2 = \frac{5 - \sqrt{5}}{10}.$$

$$\therefore u_x = \frac{(5 + \sqrt{5})(1 + \sqrt{5})^x + (5 - \sqrt{5})(1 - \sqrt{5})^x}{10(2)^x}.$$

**PROBLEM.—TO FIND THE RELATION BETWEEN THE  
MEAN ANOMALY AND THE TRUE ANOMALY.**

BY PROF. JOSEPH FICKLIN, COLUMBIA, MO.

Let  $EDF$  be the orbit of the planet, having the sun in the focus at  $S$ .



Put  $a = CE$ ,  $b = CG$ , and from  $S$  as a center and a radius  $SA$  equal to  $\sqrt{ab}$ , describe the circumference of a circle; then the circle and the ellipse will contain equal areas.

At the same time that the planet departs from  $E$ , the perihelion, let a body begin to move with a uniform motion from  $A$  through the circumference  $ABH$ , and perform a whole revolution in the same time that the planet describes the ellipse. Suppose the body describing the circle to be at  $B$  when the planet is at  $D$ ; the angle  $ASB$  is the mean anomaly, and  $ESD$  the true anomaly.

Put  $x$  = the angle  $ESD$ , and  $\theta = ASB$ ; then the area of  $ASB$  is

$$\frac{\overline{AS}^2}{2} \cdot \theta = \frac{ab}{2} \cdot \theta.$$

Put  $r$  = radius-vector of the ellipse, then the area of  $ESD$  will be

$$\frac{1}{2} \int r^2 dx; \therefore \frac{a b \theta}{2} = \frac{1}{2} \int r^2 dx, \text{ or } a b \theta = \int r^2 dx \dots \dots (1)$$

The polar equation of the ellipse, the focus being at the pole, is

$$r = \frac{p}{1+e \cos x}; \therefore \int r^2 dx = p^2 \int \frac{dx}{(1+e \cos x)^2} \dots \dots (2)$$

Assume  $\int \frac{dx}{(1+e \cos x)^2} = \frac{A \sin x}{1+e \cos x} + B \int \frac{dx}{1+e \cos x}.$

Taking the differential coefficients, we have

$$\frac{1}{(1+e \cos x)^2} = \frac{A \cos x (1+e \cos x) + A e \sin^2 x}{(1+e \cos x)^2} + \frac{B(1+e \cos x)}{(1+e \cos x)^2};$$

$$\begin{aligned} \therefore 1 &= A \cos x (1+e \cos x) + A e \sin^2 x + B (1+e \cos x) \\ &= A \cos x + A e \cos^2 x + A e \sin^2 x + B + B e \cos x = (A + B e) \cos x + A e + B. \end{aligned}$$

Equating like powers of  $(\cos x)$ , we have  $A + B e = 0$ , and  $A e + B = 1$ ; whence

$$B = \frac{1}{1-e^2}, \text{ and } A = -\frac{e}{1-e^2};$$

$$\therefore \int \frac{dx}{(1+e \cos x)^2} = -\frac{e}{1-e^2} \cdot \frac{\sin x}{1+e \cos x} + \frac{1}{1-e^2} \int \frac{dx}{1+e \cos x}.$$

Put  $y = \cos x$ ; then

$$dx = -\frac{dy}{\sqrt{1-y^2}}, \text{ and } \int \frac{dx}{1+e \cos x} = -\int \frac{dy}{(1+e y) \sqrt{1-y^2}}$$

Again, put  $1-y^2 = (1-z)^2 z^2$ ;

$$\therefore y = \frac{z^2-1}{z^2+1}, \quad dy = \frac{4z dz}{(z^2+1)^2}; \therefore -\int \frac{dy}{(1+e y) \sqrt{1-y^2}}$$

$$\begin{aligned} &= -2 \int \frac{dz}{1-e+e z^2 (1+e)} = -\frac{2}{1-e} \int \frac{dz}{1+\left(\frac{1+e}{1-e}\right) z^2} \\ &= -\frac{2}{(1-e) \sqrt{\frac{1+e}{1-e}}} \int \frac{\sqrt{\frac{1+e}{1-e}} \cdot dz}{1+\left(\sqrt{\frac{1+e}{1-e}} \cdot z\right)^2} \end{aligned}$$

$$\begin{aligned}
 &= -\frac{2}{\sqrt{(1+e)(1-e)}} \int \frac{\sqrt{\frac{1+e}{1-e}} \cdot dz}{1 + \left( \sqrt{\frac{1+e}{1-e}} \cdot z \right)^2} \\
 &= -\frac{2}{\sqrt{1-e^2}} \int \frac{\sqrt{\frac{1+e}{1-e}} \cdot dz}{1 + \left( \sqrt{\frac{1+e}{1-e}} \cdot z \right)^2} = -\frac{2}{\sqrt{1-e^2}} \cdot \tan^{-1} \left( \sqrt{\frac{1+e}{1-e}} \cdot z \right)
 \end{aligned}$$

∴ area  $ESD$

$$\begin{aligned}
 &= \frac{1}{2} \int r^2 dx = -\frac{p^2 e \sin x}{2(1-e^2)(1+e \cos x)} - \frac{2 p^2 \cdot \tan^{-1} \left( \sqrt{\frac{1+e}{1-e}} \cdot z \right)}{2(1-e^2)\sqrt{1-e^2}} + C \\
 &= -\frac{p^2 e \sin x}{2(1-e^2)(1+e \cos x)} - \frac{p^2}{(1-e^2)^{\frac{3}{2}}} \cdot \tan^{-1} \left( \sqrt{\frac{1+e}{1-e}} \cdot z \right) + C \\
 &= -\frac{p^2 e \sin x}{2(1-e^2)(1+e \cos x)} - \frac{p^2}{(1-e^2)^{\frac{3}{2}}} \cdot \tan^{-1} \left( \sqrt{\frac{1+e}{1-e}} \cdot \frac{1+y}{1-y} \right) + C \\
 &= -\frac{p^2 e \sin x}{2(1-e^2)(1+e \cos x)} - \frac{p^2}{(1-e^2)^{\frac{3}{2}}} \cdot \tan^{-1} \sqrt{\frac{1+e}{1-e} \cdot \frac{1+\cos x}{1-\cos x}} + C \\
 &= -\frac{p^2 e \sin x}{2(1-e^2)(1+e \cos x)} - \frac{p^2}{(1-e^2)^{\frac{3}{2}}} \cdot \tan^{-1} \sqrt{\frac{1+e}{1-e} \cdot \frac{2 \cos^2 \frac{1}{2} x}{2 \sin^2 \frac{1}{2} x}} + C \\
 &= -\frac{p^2 e \sin x}{2(1-e^2)(1+e \cos x)} - \frac{p^2}{(1-e^2)^{\frac{3}{2}}} \cdot \tan^{-1} \left( \sqrt{\frac{1+e}{1-e}} \cdot \cot \frac{1}{2} x \right) + C
 \end{aligned}$$

When  $x=0$ , area  $ESD=0$ ; ∴ from this equation

$$0 = -\frac{p^2}{(1-e^2)^{\frac{3}{2}}} \cdot \frac{\pi}{2} + C; \text{ whence } C = \frac{p^2 \pi}{2(1-e^2)^{\frac{3}{2}}},$$

and, therefore, the general expression for the area  $ESD$  is

$$\begin{aligned}
 \frac{1}{2} \int r^2 dx = & -\frac{p^2 e \sin x}{2(1-e^2)(1+e \cos x)} - \frac{p^2}{(1-e^2)^{\frac{3}{2}}} \cdot \tan^{-1} \\
 & \left( \sqrt{\frac{1+e}{1-e}} \cot \frac{1}{2} x \right) + \frac{p^2 \pi}{2(1-e^2)^{\frac{3}{2}}}
 \end{aligned}$$

Substituting this value of  $\int r^2 dx$  in (1) we obtain

$$\begin{aligned}
 a b \theta = & \frac{p^2 \pi}{(1-e^2)^{\frac{3}{2}}} - \frac{p^2 e \sin x}{(1-e^2)(1+e \cos x)} - \frac{2 p^2}{(1-e^2)^{\frac{3}{2}}} \cdot \tan^{-1} \\
 & \left( \sqrt{\frac{1+e}{1-e}} \cdot \cot \frac{1}{2} x \right);
 \end{aligned}$$

but  $p^2 = \frac{b^4}{a^2}$ , and  $1 - e^2 = \frac{b^2}{a^2}$ , or,  $(1 - e^2)^{\frac{3}{2}} = \frac{b^3}{a^3}$ ;  $\therefore \frac{p^2}{(1 - e^2)^{\frac{3}{2}}} = \frac{b^4}{a^2} \div \frac{b^3}{a^3}$

$= a b$ , and the above equation becomes

$$a b \theta = a b \pi - \frac{b^2 e \sin x}{1 + e \cos x} - a b \tan^{-1} \left( \sqrt{\frac{1+e}{1-e}} \cdot \cot \frac{1}{2} x \right),$$

$$\text{or, } a \theta = a \pi - \frac{b e \sin x}{1 + e \cos x} - a \tan^{-1} \left( \sqrt{\frac{1+e}{1-e}} \cdot \cot \frac{1}{2} x \right),$$

$$\text{or, } \theta = \pi - \frac{b e \sin x}{a(1 + e \cos x)} - \tan^{-1} \left( \sqrt{\frac{1+e}{1-e}} \cdot \cot \frac{1}{2} x \right) \dots \dots \dots (3);$$

and this is the relation existing between the mean anomaly and the true anomaly.

### SOLUTION OF A QUESTION INVOLVING A MINIMUM.

The following problem is found in the *Mathematical Monthly*, (Runkle's):

“Find a point,  $O$ , within a triangle, such that  $OA^n + OB^n + OC^n = a$  minimum.”

[The solution there given is a good one; but a friend of mine, in the United States Coast Survey, dreamed out the solution below, and conveyed it to paper in the morning. I give it for publication; but he disallows the use of his name.]

[Notation seen from the figure.]

$$x^n + y^n + z^n = a \text{ minimum.}$$

Let  $x$  at first be constant,  $y$  and  $z$  vary.

$$\therefore y^{n-1} dy + z^{n-1} dz = 0. \quad (1)$$

Now  $dy = x d\theta \sin \gamma$ , from infinitesimal triangle

$$\text{similarly } dz = -x d\theta \sin \beta.$$

Substituting in (1)

$$x d\theta (y^{n-1} \sin \gamma - z^{n-1} \sin \beta) = 0.$$

$$\therefore \frac{y^{n-1}}{\sin \beta} = \frac{z^{n-1}}{\sin \gamma}.$$

$$\text{If } x \text{ also vary we have } \frac{x^{n-1}}{\sin \alpha} = \frac{y^{n-1}}{\sin \beta} = \frac{z^{n-1}}{\sin \gamma}$$

If  $n=1$ ,  $\alpha=\beta=\gamma$ , each  $= 120^\circ$ .

$$\text{If } n=2, \frac{x}{\sin \alpha} = \frac{y}{\sin \beta} = \frac{z}{\sin \gamma},$$

that is, the point is the center of gravity of the triangle.

THEO. L. DE LAND, TREAS. DEP'T, WASHINGTON, D. C.

